INCOMPLETE WEIGHING DESIGNS THROUGH BALANCED TERNARY DESIGNS

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Balanced Ternary Designs (B.T.D.)

Tocher (1952) considered designs where when any treatment can occur more than once in a block of an incomplete block design. Thus the incidence matrix of an n-ary design consists of 'n' different integers as elements. Such a design, with equal block size and equal replications, is said to be balanced if the sum of products of corresponding elements in any two columns of its incidence matrix is constant. The authors developed a systematic method for obtaining balanced n-ary designs, with the above definition, by associating two suitable B.I.B. designs.

The B.I.B. designs are so chosen that the number of blocks of the first design (say b_1) is equal to the number of treatments (say v_2) of the second. When b_1, v_1, r_1, k_1, λ_1 and b_2, v_2 $(=b_1)$, r_2, λ_2 are the parameters of the respective B.I.B. designs with incidence matrices N_1 $(=b_1 \times v_1)$ and N_2 $(=b_2 \times v_2)$, the balanced n-ary design results from $N_2.N_1$ $(=N)$ with the products of any two of its different column vectors equal to $r_1^2\lambda_2+\lambda_1(r_2-\lambda_2)$. It is evident that N is of the order $b_2 \times v_1$.

From the above method of obtaining the balanced n-ary designs, it is clear that the elements of the design matrix of the resulting n-ary design depend on the arrangement of the rows and columns of the incidence matrices involved. It will be evident that the arrangement of rows of N_1 materially alters the composition of N, whereas a random arrangement of rows of N_2 disturbs only the order of the rows in N.

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If the two B I.B. designs involved in obtaining an n-ary design happen to be the same symmetric B.I.B. design belonging to either of the series (1) $v=b=4\lambda+3$, (v being prime), $r=k=2\lambda+1$, and (*ii*) $v=b=s^2+s+1$, $r=k=s+1$, $\lambda=1$, with a proper choice of the initial block, a balanced ternary design can be obtained. In this case the incidence matrix of the ternary design $N=N_1 . N_1$ ($\neq N_1 \; N_1$) where N_1 is the incidence matrix of the symmetric design belonging to either series mentioned above.

WEIGHING DESIGNS

Introduction :

Though the advantage of weighing light objects together in a chemical balance was first envisaged by Yates, Hotelling (1944) studied the problem in detail and suggested some efficient weighing designs and explored the conditions under which these designs could be efficient. Since then, Kishen (1945), Mood (1946), Banerjee (1950), Raghavarao (1959, 64), Bhaskar Rao (1966) and others have worked on these designs. B.I.B. designs for providing schemes for weighing designs have also found place in the literature. In the present investigation "balanced ternary designs" have been used to get schemes for weighing designs suitable for a chemical balance with no bias.

Though several methods of obtaining weighing designs are available in the literature, this paper attempts to obtain weighing designs wherein each weighing, in general, includes some of the objects only. Such designs have been called, by the authors, "incomplete weighing designs" and (n, s, λ) designs by Bhaskar Rao (2). Bhaskar Rao (2) showed that the necessary condition for the existence of a (n, s, λ) design, is that $n-s+(n-1)\lambda$ should be a perfect square where n is the number of rows, s is the number of zeros in each column and λ is the product of corresponding elements in any two columns of the design matrix.

Construction :

N_1^2 gives the incidence matrix of the balanced ternary design with elements 0, $\lambda, \lambda+1$ occurring once, $2\lambda+1, 2\lambda+1$ times respectively in each row or with frequencies 2, 1, 0 appearing $s\,(s+1)/2$, $(s+1)$, $s\,(s-1)/2$ times respectively in each row according as N_1 is the incidence matrix of B.I.B. design with parameters v (prime) $=b=4\lambda+3$,

$r=k=2\lambda+1$, λ or with parameters $v=b=s^2+s+1, r=k=s+1$, $\lambda=1$. Replacement of the three frequencies λ, 0, $\lambda+1$, by -1, 0, 1 or the elements 2, 1, 0 by -1, 0, 1 respectively of the balanced ternary design results in the design matrix of a balanced weighing design. We may call these weighing designs first and second series. However, the first series can be got through Hadamard matrices also.

It is easily seen that by replacing λ, $\lambda+1$ by -1, 1 and keeping 0 as it is and by replacing λ by 0, $\lambda+1$ by 1 and 0 by -1 we get two series of weighing schemes differing in the number of objects to be included in each weighing.

Estimation of weights and their variances

Subject to the following conditions, the weighing schemes for a chemical balance, with zero bias, are considered. (1) The variances of the estimated weights are equal and (2) the estimated weights are equally correlated.

Let the model for estimating weights be

$$Y=XW+e,$$

where

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_b \end{bmatrix}, \quad X=(x_1\, x_2 \,..\, x_v) = \begin{bmatrix} x_{11} & x_{12} & ..\, x_{1v} \\ x_{21} & x_{22} & ..\, x_{2v} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_{b1} & x_{b2} & ..\, x_{bv} \end{bmatrix}, \quad W=\begin{bmatrix} W_1 \\ W_2 \\ \cdot \\ \cdot \\ W_v \end{bmatrix}$$

and

$$e=\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_b \end{bmatrix}$$

where

y_i is a known weight placed to balance the pans in the ith weighing,

x_{ij} take values 1, -1, or 0 according as the jth object is placed in the left pan, right pan or not included in the ith weighing,

W_j be the true weight of the jth object to be estimated and e_i be the error associated with y_i, with expectation Zero and variance σ^2. These e_i's are assumed to be normally distributed and are independent. Further, (x_{ij}) is the incidence matrix of a balanced weighing design for weighing v objects in b weighings.

If $(X'X)$ is non-singular and (C_{ij}) its inverse, then the best estimate of W is

$$\hat{W}=(X'X)^{-1}X'.Y.$$

We can easily see that

$$\hat{W}_j=(C_{jj}-C_{ij})T_j+C_{ij}\sum_{m=1}^{v} Tm$$

with a variance $C_{jj}\sigma^2$ and covariance between any two estimated weights is $C_{ij}\sigma 2$ while T_j means $X^1{}_j\ Y$.

Efficiency

With the above notation, according to Kishen (4), the efficiency of a given design against a minimum variance design may be defined as

$$\frac{V}{b\sum_j C_{jj}}$$

and this has been followed in this paper.

The two series of weighing designs and their efficiencies

First series :

It was mentioned earlier that two balanced weighing schemes can be obtained by suitably transforming the elements of the balanced ternary design (B.T.D.) obtainable from B.I.B. designs with parameters

$$v=b=4\lambda+3, r=k=2\lambda+1, \lambda.$$

(*a*) Let $\lambda, \lambda+1$ be replaced by $-1, 1$ so that $4\lambda+2$ objects are weighed in each of $4\lambda+3$ weighings with equal number of objects included in each pan. With this scheme, $(X'X)$ becomes singular and hence the weights cannot be estimated. Bhaskar Rao (2) has also shown $(4\lambda+3, 1, 0)$ and $(4\lambda+3, 0, 0)$ designs do not exist. If in this scheme, an additional weighing with all the objects in the left pan is made, the augmented design so obtained, $(4\lambda+4, 1, 0)$ exists.

Then $$C_{jj}=\frac{1}{4\lambda+3} \text{ and } C_{ij}=0$$

and hence the efficiency

$$=\frac{4\lambda+3}{4\lambda+4}$$

This seems to be the optimum design in this class.

(*b*) Instead, in the incidence matrix of the above B.T.D., by replacing λ (or $\lambda+1$) by 0, $\lambda+1$ (or λ) *by* 1 and 0 by -1 another weighing scheme is possible with $2\lambda+2$ objects being included in each of $4\lambda+3$ weighings and always having only one object in the right pan. In this $(4\lambda+3, 2\lambda+1, \lambda-1)$ design.

$$C_{jj}=\frac{4\lambda^2-\lambda+1}{4\lambda^2(\lambda+3)} \qquad \text{And} \quad C_{ij}=\frac{-(\lambda-1)}{4\lambda^2(\lambda+3)}$$

The efficiency of this design works out to be

$$\frac{4\lambda^2(\lambda+3)}{(4\lambda+3)(4\lambda^2-\lambda+1)}$$

Second series

By replacing the three frequencies 2, 1, 0 of the incidence matrix of the B.T.D., obtainable from B.I.B. designs with

$$v=b=s^2+s=1,\ r=k=s+1,\ \lambda=1, \text{ by } 1, 0, -1$$

respectively another series of a weighing scheme can be got. It may be seen that, in this scheme, s^2-objects are weighed with $s(s+1)/2$ objects in the left pan and $s(s-1)/2$ in the right pan in each of s^2+s+1 weighings.

In these $(s^2+s+1, s+1, 0)$-designs,

$$C_{jj}=\frac{1}{s^2} \qquad \text{and} \quad C_{ij}=0$$

The efficiency works out to be $\dfrac{s^2}{s^2+s+1}$.

Comparison with some other existing designs

S_N matrices. [Raghavarao (6)]—For the existence of an optimum S_N matrices $N=P^h+1$ where P is an odd prime and h is an integer such that $P^h=1$ (mod 4) should be satisfied.

We find that for $N=4\lambda+3$ or s^2+s+1 the above condition is not satisfied. Hence a comparison is not possible.

P_N matrices. [Raghavarao (6)]—A necessary condition for their existence is $N=(d^2+1)/2$ or $2N-1$ should be a perfect square, where d is an odd integer. Only for a value $s=3$ in the second series the condition is satisfied. Hence the efficiency of this weighing scheme is compared with P_{13} matrix. The relative efficiency of (13, 4, 0) against P_{13} comes out to be 18/25, if the error variances in both cases are considered to be same.

SUMMARY

Schemes for weighing objects in a chemical balance with zero bias have been presented through the two series of balanced ternary designs. The estimation of the weights has been presented along with the efficiencies of these designs.

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EXAMPLES

First series

(*a*) Augmented design

$$(12, 1, 0)=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

Variance of each estimated weight $=\sigma^2/11$

Covariance of each pair of estimated weights $=0$

Efficiency $=11/12$.

(*b*)

$$(11, 5, 1) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Variance of each estimated weight $= \frac{3}{16}\sigma^2$

Covariance of each pair of estimated weights $=-\frac{\sigma^2}{80}$

Efficiency $=16/33$.

Second series

$$(13, 4, 0)=\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1\\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1\\
-1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1\\
1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1\\
1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & -1\\
-1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1\\
-1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0\\
0 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 0\\
0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1\\
1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & 0 & 1\\
1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & 0\\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1\\
1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 & 1 & 0
\end{bmatrix}$$

Variance of each estimated weight $= \sigma^2/_9$

Covariance of each pair of estimated weights$=0$

Efficiency$=9/13$.

$$P_{13}=\begin{bmatrix}
-1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1\\
1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1\\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1\\
1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1\\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1\\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1\\
1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1\\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1\\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1\\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1\\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1\\
1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1\\
-1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1
\end{bmatrix}$$

Variance of each estimated weight$=2\sigma^2/25$

Covariance of each pair of estimated weights$=-\sigma^2/300$

Efficiency$=25/26$.

APPENDIX

First series of weighing designs

Let N_1 be the incidence matrix of the B.I.B. design with v (prime) $=b=4\lambda+3$, $r=k=2\lambda+1$, λ developed from the initial block obtained as the even powers of the primitive element x mod. v. Then $N=N_1^2$ is the incidence matrix.

Proof :

If R_i and C_j represent the ith row and jth column of N_1, then N_1^2 contains elements $nij=R_i.C_j$. Let us consider the elements of the 1st row of N_1^2.

R_1 contains unit frequencies in positions corresponding to treatments x^{2s} $(0\leqslant s\leqslant 2\lambda)$ and zero elsewhere since the elements of the initial block of the B.I.B.D. are of the form x^{2s}. C_j will contain unities in places corresponding to the blocks in which the jth treatment occurs and zero eleswhere. Now that $n_{ij}=R_i.C_j$ is the frequency of occurrence of jth treatment in blocks bearing numbers x^{2r} $(0\leqslant r\leqslant 2\lambda)$ of the B.I.B.D. with the incidence matrix N_1.

Evidently, the contents of the block x^{2r} are of the form $x^{2s}+(x^{2r}-1)$ $(0\leqslant r,\ s\leqslant 2\lambda)$. We are interested in these treatments $j\equiv(x^{2s}+x^{2r}-1)$ mod. v.

Obviously,

$$x^{2s}+(x^{2r}-1)\not\equiv 4\lambda+2 \text{ mod. } v. \text{ for any } r \ \&\ s.$$

Hence the treatment $4\lambda+2$ does not occur in blocks bearing numbers x^{2r} (*i.e.* even powered blocks) and so

$$\text{for } j=4\lambda+2,\ n_{ij}=0.$$

Further, $4\lambda+2$ is also not a content of the block $4\lambda+3$ as

$$x^{2s}+4\lambda+3-1\not\equiv 4\lambda+2 \text{ mod. } v.$$

Consider any treatment $j\neq 4\lambda+2$ and they will be $4\lambda+2$ in all, out of which $2\lambda+1$ are the contents of the $4\lambda+3$th block. Each of these $4\lambda+2$ treatments, different from the treatment $4\lambda+2$, occurs λ times along with the treatment $j=4\lambda+2$ and is replicated $2\lambda+1$ times in the B I.B. design.

Hence any treatment, $j \neq 4\lambda+2$ which is also a content of the $4\lambda+3$th block occurs in $2\lambda+1-\lambda-1=\lambda$ of the even powered blocks, implying, thereby, for such j

$$n_{ij}=\lambda$$

and there are obviously $2\lambda+1$ such treatments.

On the other hand any treatment not being a content of the block $4\lambda+3$ and different from $4\lambda+2$, occurs in $2\lambda+1-\lambda=\lambda+1$ of the blocks bearing numbers x^{2r}.

So for all j different from $4\lambda+2$ and also not being a content of block $4\lambda+3$.

$$n_{ij}=\lambda+1$$

and they are also $2\lambda+1$ in number.

Therefore, the first row of N contains the three frequencies λ (for $j \neq 4\lambda+2$ and $j \equiv x^{2s}+4\lambda+2$ mod. v), $\lambda+1$ (for $j \neq 4\lambda+2$ and also $\not\equiv x^{2s}+4\lambda+2$ mod. v) each occurring $2\lambda+1$ times and one zero (for $j=4\lambda+2$).

Since the elements in a block of a B.I.B.D. are obtained by adding unity to the elements of the preceding block, the elements of any row of the incidence matrix N are obtained by cyclically shifting the frequencies of the previous row by one place to the right and hence N is the design matrix of the B.T.D.

Transformation of the frequencies

As the incidence matrix of the above B.T.D. is cyclic, each column contains the same frequencies as the rows (Number of rows and number of columns are same).

Further, the frequency λ in the first row of the above incidence matrix corresponds to treatments, $j \neq 4\lambda+2$ and of the form $x^{2s}+4\lambda+2$, which are contents of the block $4\lambda+3$ of the B.I.B.D. while the frequency $\lambda+1$ occurs against those treatments, $j \neq 4\lambda+2$ and $j \neq x^{2s}+4\lambda+2$, which can be shown to form the $4\lambda+3$th block of its dual.

Now, let us consider the sum of products of the corresponding elements in any two columns, say j and m, for the incidence matrix of the above B.T.D. after transforming the frequencies 0, λ and $\lambda+1$ to p, q and t respectively.

Then,

$$\Sigma\, n_{im} n_{ij} \text{ (after transformation)}$$
$$=\lambda\,(q^2+t^2)+pq+pt+(b-2\lambda-2)\,qt.$$
$$=\lambda\,(q+t)^2+pq+pt+qt \text{ (since } b=4\lambda+3).$$

The above remains constant for given values of p, q and t. Hence the design is still a balanced Ternary design. Either by putting $p=0$, $q=1$ and $t=-1$ or by putting $p=-1$, $q=1$ and $t=0$, two sets of incomplete weighing designs can be got.

Second series of weighing designs

If $A\ (=a_{ij})$ is the incidence matrix of the B.I.B.D. with $v=b=s^2+s+1, r=k=s+1, \lambda=1$ (where s is a prime or prime power) then that of the n-ary design is $N\ (=n_{ij})=A^2$.

Let the set $x_0, x_1, \ldots\ldots, x_s$ form the initial block and $x_0=1$.

If R_1 and C_j represent the first row and jth column of (a_{ij}), then $n_{ij}=R_1.C_j$—any element of the first row of $N\ (=A^2)$—is the frequency of the occurrence of jth treatment in blocks $i\equiv x_0$, x_1, $x_2 \ldots$, x_s mod. v.

The contents of the block x_k $(0\leqslant k\leqslant s)$ are of the form x_l+x_k-1 $(0\leqslant l\leqslant s)$. It is evident that the treatments of the form $2x_k-1$ occur in the blocks x_k while the treatments of the form x_l+x_k-1, $k\neq l$, occur in the two blocks x_k and x_l. There are $s+1$ treatments of the form $2x_k-1$ and $s\,(s+1)/2$ treatments of the form x_l+x_m-1 $(1\neq m)$. These two sets are mutually exclusive since $2x_k-1\not\equiv x_l+x_m-1$ mod. v as $\lambda=1$.

Thus

$$n_{ij}=1 \text{ for } j\equiv 2x_k-1 \text{ mod. } v.$$
$$=2 \text{ for } j\equiv x_l+x_m-1 \text{ mod. } v,\ 1\neq m$$
$$=0 \text{ for } j\not\equiv 2x_k-1\ \&\ j\not\equiv x_l+x_m-1\ (0\leqslant k,\ l,\ m\leqslant s)$$

There will be $\frac{s(s-1)}{2}$ $(=s^2+s+1-s\,(s+1)/2-s-1)$ zero frequencies in the first row of A^2.

Hence the first row of A^2 contains $s+1$ unit frequencies against each treatment of the form $2x_k-1$, frequency 2 against each of $s\,(s+1)/2$ treatments of the form x_l+x_m-1, $1\neq m$ and $s\,(s-1)/2$ zero frequencies against the remaining treatments.

It can also be shown that the set $2x_0, 2x_1, \ldots\ldots, 2x_s$ containing $s+1$ elements corresponding to treatments $j \equiv 2x_k - 1$ mod. v. $(0 \leqslant k \leqslant s)$ forms the block of the original B.I.B D.

By arguing in a similar way as in case of the earlier series, A^2 can be shown to be the incidence matrix of the ternary design and the columns contain in the same frequencies as rows.

Transformation of the frequencies

If we consider the product of the corresponding elements in any two columns, say j and m, of the B.T.D. which will be equal to $s(s+3)+1$, the following possible different types of products are obtained.

Type of the Product	*Its frequency in the product*	*Its transformed product*
(1)	(2)	(3)
00	x_1	pp
01	x_2	pq
02	x_3	pt
10	x_4	qp
11	λ	qq
12	x_5	qt
20	x_6	tp
21	x_7	tq
22	x_8	tt

Put $s(s-1)/2$ as r_0, $s+1$ as r and $s(s+1)/2$ as r_2. It may be remembered that the frequency 1 in A^2 reproduces the original B.I.B.D. while the other two combined, its complementary. Hence we may write and solve the following equations.

$$x_1+x_2+x_3=r_0$$

$$x_1+x_4+x_6=r_0$$

$$x_2+x_7+\lambda=r \qquad (\lambda=1)$$

$$x_4+\lambda+x_5=r$$

$$x_3+x_5+x_8=r_2$$

$$x_6+x_7+x_8=r_2$$

$$x_1+x_3+x_6+x_8=\lambda^1 \quad (\lambda^1=b-2r+\lambda)$$

$$2x_5+2x_7+4x_8+\lambda=\sum_i n_{ij}n_{im}=s(s+3)+1$$

to get

$$x_3+x_6=r_0=x_1+x_8$$
$$x_2+x_4=2x_8-r_0.$$
$$x_5+x_7=2\ (r-1)-2x_8+r_0.$$

If we replace the frequencies 0, 1, 2 in the above B.T D. by p, q, t respectively, with new types of products formed being shown at column 3 above, and if the transformed ternary design is still to be balanced, the products of any two columns of the new incidence matrix should be constant.

It means,

$$x_1p^2+(x_2+x_4)\,pq+(x_3+x_6)\,pt+q^2+(x_5+x_7)\,qt+x_8t^2$$

which can be written as,

$(r_0-x_8)\,p^2+(x_2+x_4)\,pq+(x_3+x_6)\,pt+q^2+(x_5+x_7)\,qt+x_8t^2$ should be constant. This will be so, for given values of p, q, t if $-p^2+2pq+t^2-2qt=0$ or $2q=p+t$, since $p\neq t$. Hence for p, q, t satisfying the above condition, we can get a series of balanced weighing designs.